Section 12.3: the dot product
Suppose a point $P$, while moving along a line $\ell$, experiences a constant force $F$.

- Decompose $F$ in two components $F_a$ in the direction of $\ell$ and $F_n$ perpendicular to $\ell$.
- The direction and length of $F_a$ depends on the length of $F$ and the angle $\theta$ between $F$ and $\ell$ as follows:
  \[ F_a = \cos \theta |F| \hat{v} \]
  where $\hat{v}$ is the unit vector in the direction of $\ell$.

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**Definition – geometrical**

Let $u$ and $v$ be two vectors in $\mathbb{R}^n$. The **dot product** $u \cdot v$ of $u$ and $v$ is defined as

\[ u \cdot v = |u| |v| \cos \theta \]

where $\theta$ is the angle between $u$ and $v$.

- In Dutch the dot product is called **inproduct** or **inwendig product**.
- In the previous slide the dot product of $F$ and $\hat{v}$ is
  \[ F \cdot \hat{v} = |F| |\hat{v}| \cos \theta = |F| \cos \theta, \]
  since $\hat{v}$ is a unit vector.
- Hence $F_a = (F \cdot \hat{v})\hat{v}$. 
The dot product – algebra

**Definition – algebrical**

Let \( \mathbf{u} \) and \( \mathbf{v} \) be the component form of two vectors in \( \mathbb{R}^n \):

\[
\mathbf{u} = \langle u_1, \ldots, u_n \rangle \quad \text{and} \quad \mathbf{v} = \langle v_1, \ldots, v_n \rangle
\]

The **dot product** \( \mathbf{u} \cdot \mathbf{v} \) of \( \mathbf{u} \) and \( \mathbf{v} \) is defined as

\[
\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.
\]

**Theorem**

The geometrical and algebrical definitions of the dot product are equivalent.

- The proof for \( n = 3 \) uses the Law of Cosines. See also *Thomas’ Calculus*, page 693.

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The dot product and length

**Theorem**

For all vectors \( \mathbf{u} \) we have

\[
|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}.
\]

**Proof**

Let \( \mathbf{u} = \langle u_1, \ldots, u_n \rangle \), then

\[
\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + \cdots + u_n^2 = |\mathbf{u}|^2.
\]
Properties

Theorem

For all \( u, v, w \in \mathbb{R}^n \) and \( c \in \mathbb{R} \) the following properties hold:

1. \( u \cdot v = v \cdot u \)
2. \( (cu) \cdot v = u \cdot (cv) = c(u \cdot v) \)
3. \( u \cdot (v + w) = u \cdot v + u \cdot w \)
4. \( u \cdot u = |u|^2 \).
5. \( u \cdot 0 = 0. \)

- Properties 2 and 3 can be combined for any number of vectors:
  \[
  (c_1u_1 + \cdots + cPu_p) \cdot w = c_1u_1 \cdot w + \cdots + cPu_p \cdot w.
  \]
- From property 2 and 3 follows the following formula:
  \[
  (u + v) \cdot (w + x) = u \cdot w + u \cdot x + v \cdot w + v \cdot x.
  \]

The scalar component

Definition

Let \( u \) and \( v \) be two vectors, with \( v \neq 0 \). The **scalar component of** \( u \) **in the direction of** \( v \) **(or onto** \( v \) **)** is

\[
|u| \cos \theta = \frac{u \cdot v}{|v|}.
\]

- The scalar component is equal to

\[
|u| \cos \theta = \frac{u \cdot v}{|v|}.
\]
The angle between vectors

**Theorem**

The angle between two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as

$$\theta = \arccos\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|}\right).$$

- The **inverse cosine** or **arccosine** is the inverse of the cosine restricted to the interval $[0, \pi]$.
- The inverse cosine of $x$ is denoted as $\arccos x$:

$$\arccos: [-1, 1] \rightarrow [0, \pi]$$

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**Example**

Find the angle between $\mathbf{u} = \langle 1, -2, -2 \rangle$ and $\mathbf{v} = \langle 6, 3, 2 \rangle$.

- $|\mathbf{u}| = $.
- $|\mathbf{v}| = $.
- $\mathbf{u} \cdot \mathbf{v} = $.

From this follows

$$\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} =$$

The angle $\theta$ between $\mathbf{u}$ and $\mathbf{v}$ is

$$\theta = \arccos\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|}\right) =$$
### Definition

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$. Notation: $\mathbf{u} \perp \mathbf{v}$.

- If $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$ then the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\arccos 0 = \pi/2 = 90^\circ$.
- $0 \cdot \mathbf{v} = 0$, the zero vector is orthogonal to every vector.
- If $\mathbf{u} \perp \mathbf{v}$ then $\mathbf{v} \perp \mathbf{u}$.

### Standard unit vectors

In $\mathbb{R}^2$ we have $\mathbf{i} \perp \mathbf{j}$.

In $\mathbb{R}^3$ we have $\mathbf{i} \perp \mathbf{j}$, $\mathbf{i} \perp \mathbf{k}$ and $\mathbf{j} \perp \mathbf{k}$.

### Theorem

Standard unit vectors have length 1 and are mutually orthogonal.
Orthogonality

**Example 4**

(a) *Show that* \( \mathbf{u} = \langle 3, -2 \rangle \) *and* \( \mathbf{v} = \langle 4, 6 \rangle \) *are orthogonal.*

(b) *Show that* \( \mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} \) *and* \( \mathbf{v} = 2\mathbf{j} + 4\mathbf{k} \) *are orthogonal.*

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**Projection onto a vector**

- Let \( \mathbf{u} = \overrightarrow{PQ} \) *and* \( \mathbf{v} \) *be two vectors, assume that* \( \mathbf{v} \neq 0 \).
- The line \( \ell \) *passes through* \( P \), *has the direction of* \( \mathbf{v} \), *and* \( \theta \) *is the angle between* \( \mathbf{u} \) *and* \( \mathbf{v} \).
- The **projection of** \( \mathbf{u} \) **onto** \( \mathbf{v} \) *is the vector* \( \mathbf{w} = \overrightarrow{PR} \), *where* \( R \) *is a point on* \( \ell \) *such that* \( \overrightarrow{PR} \perp \overrightarrow{RQ} \), *hence*

\[
\mathbf{w} = \overrightarrow{PR} = \]
**Definition**

Let \( \mathbf{u} \) and \( \mathbf{v} \) be two vectors with \( \mathbf{v} \neq 0 \). The **projection of \( \mathbf{u} \) onto \( \mathbf{v} \)** is defined as

\[
\text{proj}_\mathbf{v} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}
\]

**Example**

*Find the projection of \( \mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} \) onto \( \mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k} \).*

**Orthogonal decomposition**

**Definition**

Let \( \mathbf{u} \) and \( \mathbf{v} \) be two vectors with \( \mathbf{v} \neq 0 \). The **orthogonal decomposition of \( \mathbf{u} \) along \( \mathbf{v} \)** is a pair of vectors \( \mathbf{w} \) and \( \mathbf{h} \) such that

1. \( \mathbf{u} = \mathbf{w} + \mathbf{h} \),
2. \( \mathbf{w} \) and \( \mathbf{v} \) have the same direction,
3. \( \mathbf{h} \) and \( \mathbf{v} \) are orthogonal.

- Find \( \mathbf{x} \) and \( \mathbf{h} \) with orthogonal projection:

\[
\mathbf{w} = \text{proj}_\mathbf{v} \mathbf{u} \quad \text{and} \quad \mathbf{h} = \mathbf{u} - \text{proj}_\mathbf{v} \mathbf{u}
\]
Orthogonal decomposition

Example

Find the orthogonal decomposition of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ along $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$.

- See slide 14: the projection of $\mathbf{u}$ onto $\mathbf{v}$ is $\mathbf{w} = \mathbf{proj}_v \mathbf{u} = \langle -\frac{4}{9}, \frac{8}{9}, \frac{8}{9} \rangle$.

Effective force

- If we pull a box with force $\mathbf{u}$, the effective force moving the box in the direction $\mathbf{v}$ is given by the projection of $\mathbf{u}$ onto $\mathbf{v}$:

$$\text{Effective force} = \mathbf{proj}_v \mathbf{u}$$
Definition

Work done by a constant force is the magnitude of the force multiplied by the distance over which the force acts.

- This definition is only appropriate when the force is directed along the line of motion.
- The SI unit of work is the joule (J).

Work

- If a force $\mathbf{F}$ moving an object through a displacement $\mathbf{D} = \overrightarrow{PQ}$ does not have the same direction as $\mathbf{D}$, the work is performed by the component of $\mathbf{F}$ in the direction $\mathbf{D}$.
- If $\theta$ is the angle between $\mathbf{F}$ and $\mathbf{D}$ then the work $W$ is
  \[ W = (\text{scalar component of } \mathbf{F} \text{ in direction } \mathbf{D}) \times (\text{length of } \mathbf{D}) \]
  \[ = |\mathbf{F}| \cos(\theta) |\mathbf{D}| \]
  \[ = \mathbf{F} \cdot \mathbf{D} \]
Example

Let $|\mathbf{F}| = 40 \text{ N}$, $|\mathbf{D}| = 3 \text{ m}$, and $\theta = 60^\circ$. Compute the work $W$ done by $\mathbf{F}$ acting from $P$ to $Q$.

\[ W = \]
Fraction of the light that contributes to the surface color is $\cos \theta$.

$\cos \theta = \mathbf{n} \cdot \mathbf{\hat{v}}$ where $\mathbf{\hat{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$.

Represent a complicated shape by means of triangles. For each triangle, compute the dot product of its normal vector with the vector pointing in the direction of the light source: this is a measure for the colour intensity of the triangle.